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ALGEBRAIC TOPOLOGY AND INTEGRATION THEORY

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1. *Introduction.* We give here a preliminary account of a study of that side of integration theory which is useful particularly in differential geometry and fields centering around Stokes's theorem and the like. The subject is integration over r-dimensional domains (or, more generally, over r-chains) in an n-dimensional manifold or more general space (see below); the methods are direct, and are carried out largely without using coördinate systems. There are connections with real variable theory, but the problems are in general of a different nature. The methods are based on the fundamentals of algebraic topology; in fact, the work may be looked upon as the study of algebraic topology with real numbers as coefficients.

The integral $\int_A X$ of a function X over a domain A corresponds to the "scalar" or "Kronecker" product $X \cdot A$ of a cochain X and a chain A; we shall write $X \cdot A$ in place of $\int_A X$. The values of integrals are real numbers, but could equally well be elements of a Banach space. The work is largely a matter of proving inequalities; to make this possible, Lipschitz conditions are used throughout. In particular, cells σ used for domains of integration will be images of Euclidean cells σ_0 under "Lipschitz mappings" f:

$$\text{dist}[f(p), f(q)] \leqq N \text{ dist}(p, q) \text{ for some } N.$$

The smallest possible such N is the *expansion* $\xi(f, \sigma_0)$ of f in σ_0. (Though the inequalities depend on the metric in the space, the results are independent of Lipschitz changes of metric.) With this notion, a definition of the size $|A|$ of a chain A is given; for usual domains, this is the r-dimensional volume of the domain. The integral corresponds to that of a bounded function: $|\int_A X| \leqq N|A|$ for some N. The extension to the unbounded case will not be considered here.

The most general space to which our results apply is a "Lipschitz space" R, defined as follows. It is a metric space (assumed compact for simplicity), for which there is an imbedding f into some Euclidean space E^n, both f and f^{-1} being Lipschitz. Further, $f(R)$ is supposed "locally Lipschitz connected" in each dimension r. That is, for some $\eta > 0$ and some N_r, any singular $(r-1)$-sphere S^{r-1} of diameter $< \eta$ in $f(R)$ bounds a singular r-cell Q^r in $f(R)$ such that

$$\text{diam } (Q^r) \leqq N_r \text{ diam } (S^{r-1}).$$

There is then a retraction ρ of a neighborhood U of $f(R)$ in E^n into $f(R)$ such that ρ is Lipschitz.

2. *Lipschitz chains.* By vol (σ_0^r) we mean the usual r-volume of a Euclidean simplex σ_0^r in metric Euclidean space. We define $|\sigma^r|$ for a Lipschitz singular simplex $\sigma^r = f(\sigma_0^r)$, i.e., image of a Euclidean simplex σ_0^r under a Lipschitz mapping f, as follows. Consider subdivisions $\sigma_0^r = \Sigma\sigma_{0i}^r$ of σ_0^r into Euclidean simplexes σ_{i0}^r, and affine mappings g_i of Euclidean simplexes τ_i^r into σ_{0i}^r; set $f_i(p) = f(g_i(p))$ in τ_i^r; then

$$|\sigma^r| = GLB \sum_i \xi^r(f_i, \tau_i) \text{ vol } (\tau_i),$$

taking the greatest lower bound over all such expressions. For a Euclidean simplex or smooth simplex σ, $|\sigma| = \text{vol } (\sigma)$. To extend the definition to Lipschitz singular chains $A = \Sigma a_i \sigma_i^r = \Sigma a_i f(\sigma_{0i}^r)$, consider simultaneous subdivisions $\Sigma_j \sigma_{0ij}^r$ of the σ_{0i}^r, and proceed as above, using $|a_i|$; coefficients must be collected over the singular simplexes occurring.[1]

3. *Lipschitz cochains. A Lipschitz r-cochain* X^r in R is a real-valued function $X \cdot A^r$ of Lipschitz singular chains A^r in R, with the following properties: For some numbers N, N',

(a) $X \cdot (-\sigma) = -X \cdot \sigma$,

(b) $|X \cdot \sigma| \leqq N|\sigma|$,

(c) $|X \cdot \partial\sigma| \leqq N'|\sigma|$.

Here, $\partial\sigma$ denotes the boundary of an $(r+1)$-simplex σ. We let $|X|$ and $|\delta X|$ denote the least possible values of N and N', respectively. X, defined for simplexes, becomes a linear function of Lipschitz singular r-chains by setting $X \cdot \Sigma a_i \sigma_i^r = \Sigma a_i X \cdot \sigma_i^r$. Now $|X \cdot A| \leqq |X|\,|A|$.

The *coboundary* δX of X is defined by

$$\delta X^r \cdot A^{r+1} = X^r \cdot \partial A^{r+1}.$$

Written in the form $\int_A \delta X = \int_{\partial A} X$, this becomes Stokes's theorem.

If f is a Lipschitz mapping of R into R', it carries cochains $X' = X'^r$ of R' into cochains f^*X' of R, defined by $f^*X' \cdot A = X' \cdot fA$. We have $\delta f^*X' = f^*\delta X'$, and

$$|f^*X'| \leqq \xi^r(f)\,|X'|,\ |\delta f^*X'| \leqq \xi^{r+1}(f)|\delta X'|.$$

A 0-cochain X is simply a point function: $X(p) = X \cdot p$. Its coboundary is given by $\delta X \cdot (pq) = X \cdot \partial(pq) = X \cdot q - X \cdot p$.

4. *Definition for polyhedral chains.* Let $A = fA_0$ be a Lipschitz chain in R, expressed as the image of the chain A_0 in the simplicial complex K_0. Let K_1 be a simplicial subdivision of K_0. Let $\mathfrak{S}A_0$ be a subdivision of A_0. Construct the cartesian product $I \times K_0$ of the unit interval $I = (0, 1)$ and K_0; consider K_0 as $0 \times K_0$ and K_1 as $1 \times K_0$. We may subdivide it simplicially so that

$$\partial\mathfrak{S}(I \times A_0) = \mathfrak{S}A_0 - A_0 - \mathfrak{S}(I \times \partial A_0).$$

Map $I \times K_0$ into R by setting $F(t, p) = f(p)$. Then for each τ of $\mathfrak{S}(I \times A_0)$ not in $0 \times K_0$ or $1 \times K_0$, $|F(\tau)| = 0$. It follows from (b) and (c) that

$$X \cdot \mathfrak{S}A = X \cdot F\mathfrak{S}A_0 = X \cdot FA_0 = X \cdot A.$$

Thus $X \cdot A$ is independent of subdivisions of A, and may be defined for "polyhedral singular chains" A.

Essentially the same method (with $K_1 = K_0$) shows that $X \cdot A$ is continuous under deformations of A.

For n-cochains X in E^n (or in an n-manifold) we may define $X \cdot Q$ for open sets Q; thus, X gives rise to a Lebesgue integral $X \cdot Q = \int_Q D_X(p)dp$, with the ordinary Lebesgue measure. Such a theorem is unknown for X^r in E^n, $1 < r < n$.

5. *Special cochains; the de Rham Theorem.* For an oriented n-simplex σ^n in metric oriented E^n, let $[\sigma^n]$ be $\pm$ vol (σ^n) according as σ^n and E^n are oriented alike or unlike. Write $I^n \cdot \sigma^n = [\sigma^n]$; then I^n is the *unit n-cochain* of metric oriented E^n. For any 0-cochain W^0 and r-cochain X^r, we may define the r-cochain W^0X^r by letting $(W^0X^r) \cdot \sigma^r = (W^0 \cdot p)(X^r \cdot \sigma^r)$ approximately, for p in σ^r, if σ^r is small; the definition is completed by a passage to the limit (using subdivisions of σ). Now, with a notation from §4, $X^n = D_X I^n$ in E^n.

A cochain X is a *cocycle* if $\delta X = 0$; X and Y are *cohomologous* if $Y - X = \delta Z$ for some Z. The classes of cohomologous cocycles are the elements of the rth *Lipschitz cohomology group* of the space.

In a simplicial complex K, to each Lipschitz cochain X corresponds an algebraic cochain ϕX, defined by $\phi X \cdot \sigma_i^r = X \cdot \sigma_i^r$ for each r-simplex σ_i^r of K. *This defines an isomorphism between the Lipschitz and the algebraic cohomology groups.* This is the counterpart of the de Rham Theorem[2] (originally expressed in terms of homology theory). See also §§7 and 8.

In the proof, we define for each algebraic cochain X a Lipschitz cochain ψX as follows. It is sufficient to define $\psi X \cdot \tau$ for simplexes τ lying in simplexes of K (compare §6). If we define $\psi\sigma^r$ for each σ^r, ψX for all

X will be defined. Take $\sigma^r = p_0 \ldots p_r$, and any $\tau^r = q_0 \ldots q_r$ in a simplex of K. If τ is not in the star of σ, set $\psi\sigma \cdot \tau = 0$. Otherwise, say τ is in $\sigma^s = p_0 \ldots p_r p_{r+1} \ldots p_s$. Set

$$\psi\sigma \cdot \tau = [q_0 \ldots q_r p_{r+1} \ldots p_s]/[\sigma].$$

This is independent of the (Euclidean) metric employed. We may prove: $\phi\psi X = X$, $\delta\psi X = \psi\delta X$.

6. *Lipschitz skeleton cochains.* A Lipschitz r-cochain X must have a value for all Lipschitz r-chains A; these chains form a very large class of objects. We shall show that Lipschitz cochains may be derived from "Lipschitz skeleton cochains," defined for a much more restricted class of objects.

A *skeleton r-simplex* σ^r in a metric space R is a set of $r + 1$ points $p_0, \ldots, p^r$ in R. It is oriented like an abstract simplex. A *defining set of sides* of σ^r is a set $p_{\lambda_i} p_{\mu_i}$ such that if the p_i are put in E^r, the vectors $v_i = p_{\mu_i} - p_{\lambda_i}$ $(i = 1, \ldots, r)$ are independent. The *potentiality* of σ is

$$\text{Pot } (\sigma) = \frac{1}{r!} \min \text{ dist } (p_{\lambda_1}, p_{\mu_1}) \ldots \text{dist } (p_{\lambda_r}, p_{\mu_r}),$$

taking the minimum over defining sets of sides of σ.

A *Lipschitz skeleton cochain* x is a function $x \circ \sigma$ such that (a′) $x \circ (-\sigma) = -x \circ \sigma$, and

$$\text{(b}') \quad |x \circ \sigma| \leqq N \text{ Pot } (\sigma), \quad \text{(c}') \quad |x \circ \partial\sigma| \leqq N' \text{ Pot } (\sigma),$$

for some N, N', as before. Again $x \circ \Sigma a_i \sigma_i^r = \Sigma a_i x \circ \sigma_i^r$.

We shall use a *standard* sequence of subdivisions[3] $K_1, K_2, \ldots$, of a complex K_0: only a finite number of shapes of simplexes may occur in all the K_i. If $A = fA_0$, A_0 in K_0, and $\mathfrak{S}_n A_0$ is the subdivided chain in K_n, etc., as in §4, then

$$\bar{x} \cdot A = \lim_{n \to \infty} x \circ f \mathfrak{S}_n A_0$$

exists, and $\bar{x}$ is a Lipschitz cochain X, depending on x only. Conversely, any X gives rise to an x for which $\bar{x} = X$.

Skeleton cochains x, y are *equivalent* if $\bar{x} = \bar{y}$. This holds if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|y \circ \sigma - x \circ \sigma| \leqq \epsilon \text{ Pot } (\sigma) \text{ if diam } (\sigma) < \delta.$$

7. *Tensor cochains.* For simplicity, we shall remain in E^n. A *Lipschitz tensor cochain* is a function $T(p;\ v_1, \ldots, v_r)$ of the point p and vectors $v_1, \ldots, v_r$, which (a'') is linear and alternating in the vectors, and such that

$$\text{(b}'') \quad |T(p;\ v_1, \ldots, v_r)| \leqq N |v_1| \ldots |v_r|,$$

$$\text{(c}'') \quad |T(p';\ v_1, \ldots) - T(p;\ v_1, \ldots)| \leqq N' |p' - p|\, |v_1| \ldots |v_r|.$$

We may define $T(p;\ V)$ where V is a contravariant r-vector, i.e., alternating product, $v_1 \ldots v_r$ (or a sum of such). For fixed p, T is a covariant r-vector.

For a point function $u(p)$ and a vector v, set

$$\nabla_v u(p) = \lim_{t \to 0} [u(p + tv) - u(p)]/t.$$

The covariant derivative of T is, if T is differentiable,

$$\delta T(p;\ v_1 \ldots v_{r+1}) = \sum (-1)^{i-1} \nabla_{v_i} T(p;\ v_1 \ldots \hat{v}_i \ldots v_{r+1});$$

$\hat{v}_i$ denotes that this term is to be omitted.

To any oriented Euclidean simplex $\sigma^r = p_0 \ldots p_r$ in E^n corresponds a contravariant r-vector $\{\sigma^r\}$, defined by

$$\{\sigma^r\} = \frac{1}{r!} (p_1 - p_0) \ldots (p_r - p_0);$$

if $v_1, \ldots, v_r$ is any defining set of sides of σ^r, $\{\sigma^r\} = \pm v_1 \ldots v_r$. The center of σ^r is $p_\sigma = \Sigma p_i/(r+1)$.

To each T corresponds a Lipschitz skeleton cochain $x = \Theta T$ defined by

$$\Theta T \cdot \sigma = T(p_\sigma;\ \{\sigma\}).$$

In the proof that (c′) holds and for other matters, the following combinatorial theorem (corresponding to Stokes's theorem) is needed. Let $L_0, \ldots, L_r$ be covariant r-vectors. Let $\sigma^r = p_0 \ldots p_r$ be a simplex, with a defining set of sides $v_i = p_{\mu_i} - p_{\lambda_i} (i = 1, \ldots, r)$. Let σ_i be the face of σ opposite p_i. Then

$$\sum_{i=0}^{r} (-1)^i L_i \{\sigma_i\} = \frac{1}{(r-1)!} \sum_{j=1}^{r} (-1)^j [L_{\mu_j}(\ldots \hat{v}_j \ldots) - L_{\lambda_j}(\ldots \hat{v}_j \ldots)].$$

Though $\delta\Theta T \neq \Theta\delta T$ in general, they are equivalent (if T is differentiable); hence $\delta\overline{\Theta T} = \overline{\delta\Theta T} = \overline{\Theta\delta T}$, as Lipschitz cochains. Using $\overline{\Theta T}$ defines an integration process without using coördinate systems.[4] With a coördinate system, the theory becomes ordinary integration; the theory of the Jacobian, Stokes's theorem, etc., follows simply.

The "tensor cohomology groups" again are isomorphic to the algebraic cohomology groups. (The de Rham Theorem is for the case that the tensors are differentiable.)

8. *Products of cochains.* The product o ftensor cochains is well known. The corresponding product among Lipschitz cochains satisfies the following conditions:

(α) $X^r Y^s$ is an $(r+s)$-cochain, linear in both X^r and Y^s.

(β) $X^0 Y^s$ is as defined in §5.

(γ) $|X^r Y^s| \leqq N_{rs}|X^r|\,|Y^s|$.

(δ) $\delta(X^r Y^s) = \delta X^r Y^s + (-1)^r X^r \delta X^s$.

For general Lipschitz cochains, the products with these properties exist and are uniquely determined. (This holds in a Lipschitz space.) The usual properties hold:

$$(XY)Z = X(YZ);\quad Y^s X^r = (-1)^{rs} X^r Y^s;$$

for Lipschitz mappings f, $f^*(XY) = (f^*X)(f^*Y)$.

The product corresponds (under the operation ϕ of §5) to the product[5] $X \smile Y$ of algebraic cochains. Hence the cohomology rings in the algebraic, Lipschitz and tensor cases are isomorphic, completing the de Rham Theorem for these cases.

The fundamental tool for this section is the approximation to a cochain X by a set of coboundaries Z_i. Taking R in E^n, let ρ be the retraction of §1. Cut U into small pieces Q_i; take p_i in Q_i, and set

$$Z_i \cdot \sigma^{r-1} = X \cdot \rho J(p_i, \sigma^{r-1})$$

for any σ^{r-1} in Q_i; J denotes the join. Then $|X - \delta Z_i|$ is small in Q_i.

9. *The continuity theorem.* A continuity theorem was mentioned in §4. We state a much more general (and difficult) theorem. Let A be a Lipschitz chain in R. For any $\epsilon > 0$ and N, there is an $\eta > 0$ with the following property. For any Lipschitz chain B satisfying $|B| < N$, $|\partial B| < N$, and one further condition described below,

$$|X \cdot B - X \cdot A| \leqq \text{max. } (|X|, |\delta X|)\epsilon.$$

The further condition is: there exist chains (not necessarily Lipschitz) C within η of ∂A and D within η of A such that

$$B - A = \partial D + C.$$

Letting $\eta \to 0$ is interesting: two chains A and B are "homology equivalent," i.e., C and D exist for all $\eta > 0$, if and only if $X \cdot A = X \cdot B$ for all X.

10. *Stokes's theorem.* With the help of methods described above (including the continuity theorem), a very general form of Stokes's theorem may be proved. We shall not give details here.

[1] See, for instance, Seifert-Threlfall, *Lehrbuch der Topologie*, Leipzig, 1934, Chapter IV.

[2] See G. de Rham, *Jour. de Math.* (10), **9**, 115–200 (1931).

[3] See H. Freudenthal, *Annals Math.*, **43**, 580–582 (1942).

[4] In an earlier version of this work, a large part of the theory was worked out by Paul Olum (1940, unpublished). See in this connection H. Whitney, *Duke Math. Jour.*, **4**, 495–528 (1938), §§11, 12.

[5] See, for instance, H. Whitney, *Annals Math.*, **39**, 397–432 (1938); see also H. Whitney, *Bull. Am. Math. Soc.*, **43**, 785–805 (1937), p. 803.

GEOMETRIC METHODS IN COHOMOLOGY THEORY[1]

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1. *Introduction.* In algebraic topology, the concepts of chains and cycles have considerable geometric content. Singular chains and cycles carry this content over to more general spaces such as polyhedra and absolute neighborhood retracts (*ANR*'s). Cochains and cocycles, algebraically as simple as chains and cycles, did not appear until about 1935, in part because of the lack of a geometric interpretation. We give such an interpretation here;[2] it is, we believe, the natural one. Geometric cochains will be functions of singular chains in "general position."

The theory should apply to *ANR*'s R; at present, some of the basic theorems are proved only for polyhedra P. Full details will be given elsewhere. The coefficient group H will be supposed discrete. If H is continuous, for example, if it is the group of reals, the "nuclear system" (see below) may be discarded in whole or in part.[3]

2. *r-thin subsets of R.* A subset N of R is *r-thin* in R if any singular r-cell $\sigma^r = f(\sigma_0^r)$ in R may be pulled away from N by an arbitrarily small deformation f_t; $f_1(\sigma_0^r) \cap N = 0$. For example, a k-plane in n-space E^n is $(n-k-1)$-thin in E^n. The set N and its complement may be very complicated.

Let f map the polyhedron P into P'; let N be r-thin in P'. Then there is an arbitrarily small deformation f_t of f such that $f_1^{-1}(N)$ is r-thin in P. (This has not been proved for *ANR*'s).

Let N, N' be point sets in R. The singular chain A in R is in *general position* relative to (N, N') if A and ∂A do not intersect N' and N, respectively.

3. *Geometric cochains.* A *geometric r-H-cochain* $X = X^r$ in R is a system of the following sort. It has a *nucleus* $\bar{X}$ and a *nuclear boundary* $\bar{X}' \subset \bar{X}$; $\bar{X}$ and $\bar{X}'$ are closed, and are $(r-1)$-thin and r-thin, respectively. For any singular r-chain A, with integer coefficients, in general position relative to $(\bar{X}, \bar{X}')$, $X \cdot A$ is defined, and is in H; this is a homomorphism of the group of these chains into H. Further,

(*a*) $X^r \cdot A^r = 0$ if A^r does not intersect $\bar{X}$,

(*b*) $X^r \cdot \partial A^{r+1} = 0$ if A^{r+1} does not intersect $\bar{X}'$.

The *coboundary* δX of X, with nuclear system $(\bar{X}', 0)$, is defined by $\delta X^r \cdot A^{r+1} = X^r \cdot \partial A^{r+1}$.

Clearly X is determined by its values for arbitrarily small singular cells near points of $\bar{X} - \bar{X}'$. Keeping all chains in general position, $X \cdot A$ is easily proved independent of subdivisions and deformations of A.

If $X \cdot \sigma = 0$ for all σ in some neighborhood U of p in $\bar{X}$, the points of $\bar{X}$ in U may be dropped out. In this way, we obtain the *irreducible nucleus* of X. Note now that $X \cdot A$ is always defined if $\delta X = 0$, $\partial A = 0$.

For a simple example, let $\bar{X}$ be a square in E^2; let $\bar{X}'$ be its boundary. Let u be a normal vector to $\bar{X}$. Then if A^1 is a segment cutting through $\bar{X}$, let $X^1 \cdot A^1$ be 1 or -1 according as A^1 is directed like u or unlike u. If A^2 is a disc cutting $\bar{X}'$, $\delta X^1 \cdot A^2 = X^1 \cdot \partial A^2 = \pm 1$.

Without further conditions, a cochain may be very complicated. For example, in E^n ($n \geqq 1$), $X^n \cdot \sigma^n$ may have arbitrarily large values for arbitrarily small σ^n near a given p in X.

4. *Standard cochains in a complex.* In a complex K the "dual" ξ^r_i of a cell σ^r_i of K consists of simplexes of the first derived of K, of various dimensions in general; it forms a point set which is $(r-1)$-thin in K. It forms the nucleus of a cochain $X^r_i = \phi\sigma^r_i$ such that $X^r_i \cdot \sigma^r_j = \delta_{ij}$. Setting $\phi \Sigma a_i \sigma^r_i = \Sigma a_i X^r_i$, we have $\delta\phi X = \phi\delta X$ for algebraic cochains X of K. These *standard cochains* operate on the algebraic chains of K, thought of as singular chains, just as the algebraic cochains do.

5. *Mappings and deformations.* Let f map P_1 into P_2. Then given X_2 in P_2, we may deform f into f_1 (see §2) to make $(f_1^{-1}(\bar{X}_2), f^{-1}(\bar{X}_2'))$ the nuclear system of a cochain $X_1 = f_1^* X_2$ in P_1, defined by

$$f_1^* X_2 \cdot A_1 = X_2 \cdot f_1 A_1.$$

Then $\delta f_1^* X_2 = f_1^* \delta X_2$. If $f_t(p)$ (p in P_1, $0 \leqq t \leqq 1$) is a deformation, such that $f_0^* X_2$ and $f_1^* X_2$ are defined, we may "deform" the deformation for $0 < t < 1$, and then define an $(r-1)$-cochain $f_I^* X_2$, such that

$$\delta f_I^* X_2 = f_1^* X_2 - f_0^* X_2 - f_I^* \delta X_2.$$

6. *A local property of cochains.* Given X^r in R ($r > 0$), and p in $\bar{X} - \bar{X}'$, let f_t be a deformation of the identity in R such that $f_1(p)$ is not in $\bar{X}$. Say p is in U, $f_1(U) \cap \bar{X} = 0$, and $f_t(U) \cap \bar{X}' = 0$ for all t. Now $f_0^* X = X$, $f_1^* X = 0$ in U, and $f_I^* \delta X = 0$ in U; hence, by the relation above, $X = \delta f_I^* X$ in U. Thus, *every geometric cochain of dimension > 0 is locally a coboundary.*

7. *Geometric cohomology groups in a complex.* Let X^r ($r > 0$) be a geometric cochain in the complex K such that δX is standard (§4). With methods above, and other tools, we may show that there is a standard Y and a geometric Z such that $X - Y = \delta Z$. It follows that the geometric and algebraic cohomology groups of K are isomorphic.

8. *Properties of singular chains in terms of geometric cochains.* Conditions that a cochain X be 0, or a cocycle, or a coboundary, are expressible in a simple manner in terms of $X \cdot A$ for various A; we shall state some corresponding theorems for singular chains. Say A_1 and A_2 are *essentially equal:* $A_1 = A_2$, if, for arbitrary neighborhoods U of $A_1 \cup A_2$ and U'

of $\partial A_1 \cup \partial A_2$, there are singular chains $B \subset U$ and $C \subset U'$ such that $A_1 - A_2 = \partial B + C$. (This relation is not transitive, unless some assumption to eliminate chains with space-filling characteristics is made.) Now the conditions $A =_0 0$, $\partial A =_0 0$, $A =_0 \partial B$ for cycles A, are equivalent to $X \cdot A = 0$, $\delta X \cdot A = 0$, $X \cdot A = 0$ if $\delta X = 0$, for all cochains X of the proper dimension in general position relative to A.

9. *Products of geometric cochains.* We first give an example. Let $\bar{X}$ and $\bar{Y}$ be squares in E^3, as in §3; let them intersect in a segment $\bar{Z}$. (Do not let $\bar{X}'$ and $\bar{Y}'$ intersect.) Take p in $\bar{Z}$; let u and v be vectors at p normal to $\bar{Z}$, and lying in $\bar{Y}$ and $\bar{X}$, respectively. If A^1 is a segment cutting $\bar{X}$ in the direction of u, say $X \cdot A^1 = \alpha$; define β similarly for $\bar{Y}$ and v. Then if A^2 is a disc cutting $\bar{Z}$ at p and containing u and v, and A^2 is oriented by the ordered pair (u, v), set $Z^2 \cdot A^2 = \alpha\beta$. We set

$$Z = X \smile Y;\ \bar{Z} = \bar{X} \cap \bar{Y},\ \bar{Z}' = (\bar{X} \cap \bar{Y}') \cup (\bar{X}' \cap \bar{Y}).$$

We now discuss the general situation. Say X^r and Y^s are in *general position* if $\bar{Z}$ and $\bar{Z}'$ defined above are $(r + s - 1)$-thin and $(r + s)$-thin, respectively, and $\bar{X}' \cap \bar{Y}'$ is $(r + s + 1)$-thin. (Given arbitrary X and Y, slight deformations f_t and g_t of the identity $f_0 = g_0$ will bring $f_1^* X$ and $g_1^* Y$ into general position.) Given any p in $\bar{Z} - \bar{Z}'$, define Z in a small neighborhood U of p as follows. First, if $r = 0$,

$$Z^s \cdot \sigma^s = (X^0 \cdot p)(Y^s \cdot \sigma^s),\ \sigma^s \text{ in } U.$$

For $r > 0$, say $X = \delta X_1$, $\delta Y = 0$, in U (see §6); using induction, set

$$Z^{r+s} \cdot \sigma^{r+s} = (X_1^{r-1} \smile Y^s) \cdot \partial\, \sigma^{r+s},\ \sigma^{r+s} \text{ in } U.$$

The proof of existence and uniqueness, together with the usual properties of the product, is carried out with the help of induction.

10. *The usual products in a simplicial complex.* If K is a simplicial complex with ordered vertices, there is a simple way of defining two systems of duals ξ_t^l, η_j^s (see §4) so that any corresponding $\varphi_\xi X^r$, $\varphi_\eta Y^s$ are always in general position.[4] Now for any algebraic X^r and Y^s, and any simplex σ^{r+s} of K, which we may think of as a singular simplex,

$$(\varphi_\xi X^r \smile \varphi_\eta Y^s) \cdot \sigma^{r+s}$$

is defined, and is the same as the usual algebraic product $(X \smile Y^s) \cdot \sigma^{r+s}$.

[1] Presented to the Am. Math. Society, Feb. 21, 1941, under the title "Geometric methods in combinatorial topology."

[2] For earlier work along these lines, see S. Wylie, *Proc. London Math. Soc.*, **46**, 174–198 (1940). An application will be found in H. Whitney, *Annals Math.* (2), **45**, 220–246 (1944), §5.

[3] The theory then takes on a completely different form. See the preceding paper in these PROCEEDINGS.

[4] Compare H. Whitney, these PROCEEDINGS, **26**, 143–148 (1940), §5.

COMPLEXES OF MANIFOLDS[1]

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1. *Introduction.* A smooth manifold M (smooth means continuously differentiable) is a well-defined concept. A bounded manifold may have, for boundary, anything from a smooth manifold to quite a general point set. For analytical purposes, the boundary should at least be made up of pieces of manifolds, joining together smoothly; that is, it should be formed of a "complex of manifolds," or "complifold" for short. Then $M = \sigma^n$, together with the boundary cells σ_i^r, forms a complifold K. If one tries to define K abstractly, one finds that all sorts of curious situations may occur locally. In this preliminary note, we wish to point out some of the properties that will bring K back to reasonableness. If K is imbeddable in a Euclidean space E^m, each cell being imbedded smoothly, this should be satisfactory.

2. *Complifolds.* Instead of giving a full definition, we shall point out some of the properties that are clearly necessary. The cells should attach to each other, in the point set sense, like the cells of an ordinary complex. The relation between coördinate systems in incident cells should be smooth, with Jacobian matrix of maximum rank. If a vector is tangent to faces σ_1, σ_2 of a cell σ at p, it should be tangent to a common face of σ_1 and σ_2.

Among the important subjects we cannot discuss here, we mention the following. The property of K being "locally flat" or "locally simplicial"; the property of K_1 and K_2 being in "general position" in K; the fact that, in this case, $K_1 \cap K_2$ is a complifold.

The local properties of K can be described largely in terms of tangent vectors, to which we now turn.

3. *The tangent space of K at a point.* For each (closed) cell σ, and point p in σ, there is a *tangent cone* $\Gamma(\sigma, p)$ of vectors (σ, v) which are tangent to σ at p; these generate a vector space $V(\sigma, p)$. Given p in K, let $\sigma_1, \ldots, \sigma_s$ be the cells containing p. Define (abstractly) the direct sum

$$V^*(p) = V(\sigma_1, p) \oplus \ldots \oplus V(\sigma_s, p).$$

The vectors

$$w_{ij}(v) = (\sigma_i, v) - (\sigma_j, v), v \text{ in } V(\sigma_i, p) \cap V(\sigma_j, p)$$

generate a subspace $V^{**}(p)$ of $V^*(p)$. The *tangent space*

$$V(K, p) = V^*(p) \ominus V^{**}(p)$$

of K at p is formed from $V^*(p)$ by setting all $w_{ij}(v) = 0$.

We say K is *consistent at* p if each $\Gamma(\sigma, p)$ lies in $V(K, p)$ in a one-one manner. This is obviously a necessary condition that K be smoothly imbeddable in E^m.

It may happen that a relation $a_1v_1 + \ldots + a_kv_k = 0$ is not true at p, yet arbitrarily near p a relation arbitrarily near this one holds. This might preclude smooth imbedding.

A necessary and sufficient condition for imbeddability of a general complifold seems highly difficult to obtain. Instead, we shall give a slight restriction on K.

4. *Cellwise homogeneity.* Let K be consistent at each p. If for each σ, and cells $\sigma_1, \ldots, \sigma_s$, with σ as face, the part of $V(K, p)$ over these cells is of constant dimension as p moves over σ, we say K is *cellwise homogeneous.*

THEOREM: *Any cellwise homogeneous complifold of dimension n may be smoothly imbedded in E^{2n+1}, and smoothly immersed in E^{2n}.*

[1] Presented to the American Mathematical Society Sept. 8, 1942.

ON TOTALLY POSITIVE FUNCTIONS, LAPLACE INTEGRALS AND ENTIRE FUNCTIONS OF THE LAGUERRE-POLYA-SCHUR TYPE

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The purpose of this note is to furnish a solution of the problem of representing by Laplace integrals the reciprocals of entire functions with real zeros only, which are of genus zero or one up to certain exponential factors. The solution appears in terms of a certain class of functions which we propose to call *totally positive functions.*

1. *Totally Positive Functions.*—A real-valued function $\Lambda(x)$, defined for all real x, is said to be *totally positive* (abbreviated: t. p.) if it satisfies the following three conditions:

(α) $\Lambda(x)$ is measurable.

(β) If $x_1 < x_2 < \ldots < x_n$ and $t_1 < t_2 < \ldots < t_n$, then the determinant of order n, whose element in the ith row and jth column is $\Lambda(x_i - t_j)$, should be non-negative, i.e.,

$$\det\|\Lambda(x_i - t_j)\| \geqq 0. \tag{1}$$

This inequality should be verified for $n = 1, 2, 3, \ldots$.

(γ) $\Lambda(x)$ should be positive for at least two distinct values of x.

For $n = 1$, the inequality (1) is equivalent to $\Lambda(x) \geqq 0$ for all real x, justifying the name of total positivity for our set of conditions.

A trivial example of t. p. functions is furnished by

$$\Lambda(x) = e^{ax+b}. \tag{2}$$

Indeed, all determinants (1) vanish if $n \geqq 2$. Another t. p. function is

$$\Lambda(x) = \begin{cases} e^{-x} & \text{if } x \geqq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{3}$$

Indeed, we readily find in this case that[1]

$$\det\|\Lambda(x_i - t_j)\| = \exp\{-\Sigma x_i + \Sigma t_j\} \cdot \epsilon,$$

where $\epsilon = 1$ or $\epsilon = 0$, depending on whether or not all inequalities $t_1 \leqq x_1 < t_2 \leqq x_2 < t_3 \leqq x_3 < \ldots < t_{n-1} \leqq x_{n-1} < t_n \leqq x_n$ hold simultaneously. Incidentally, with $\Lambda(x)$ defined by (3), the functions $\Lambda(ax + b)$ $(a \neq 0)$ are the only discontinuous t. p. functions. A further noteworthy example is

$$\Lambda(x) = e^{-x^2}. \tag{4}$$

The total positivity of this function is likewise clear because

$$\det\|\exp\{-(x_i - t_j)^2\}\| = \exp\{-\Sigma x_i^2 - \Sigma t_j^2\} \cdot \det\|\exp\{2x_i t_j\}\|,$$

the last determinant being known to be positive.[2]

It is easy to see that of the conditions (α), (β), (γ), neither one is implied by the other two. Thus, if

$$\Lambda(x) = \exp\{-\psi(x)\}, \tag{5}$$

where $\psi(x)$ is a discontinuous, hence non-measurable, solution of $\psi(x + y) = \psi(x) + \psi(y)$, then the conditions (β) and (γ) are fulfilled without (α) holding.

The conditions (α) and (γ), together with the inequality (1), for $n = 1$ and $n = 2$ only, are found to be equivalent with the requirement that the function $\psi(x)$, of (5), is a convex function. This remark readily leads to the following proposition: *For any t. p. function* $\Lambda(x)$ *we have*

$$0 < \int_{-\infty}^{\infty} \Lambda(x)dx \leqq +\infty.$$

A t. p. function $\Lambda(x)$ *is either monotone (in the wide sense), in which case*

$$\int_{-\infty}^{\infty} \Lambda(x)dx = +\infty, \tag{6}$$

or else it is not a monotone function, in which case we have

$$0 < \int_{-\infty}^{\infty} \Lambda(x)dx < +\infty. \tag{7}$$

A non-monotone t. p. function $\Lambda(x)$ may therefore also be thought of as a frequency function. We wish to call them *Polya frequency functions.* Among our examples of t. p. functions we find that (2) is monotone, while

(3) and (4) are Polya frequency functions. A study of the class of t. p. functions reduces essentially to a study of its subclass of Polya frequency functions in view of the following proposition: *If* $\Lambda(x)$ *is t. p. then either* $\Lambda(x) = \exp\{ax + b\}$, *or else there is a number* ω *such that* $\Lambda_0(x) = e^{\omega x} \cdot \Lambda(x)$ *is a Polya frequency function.*

2. *Entire functions of the Laguerre-Polya-Schur type.*—Following Pólya and Schur[3] we shall say that an entire rational or transcendental function $\Phi(z)$ is an *entire function of type I*, if its canonical representation is of the form

$$\Phi(z) = Cz^n e^{\gamma z} \prod_{\nu=1}^{\infty} (1 + \delta_\nu z), \quad (C \gtrless 0,\ n \geqq 0,\ \gamma \geqq 0,\ \delta_\nu \geqq 0). \tag{8}$$

Likewise we shall say that $\Psi(z)$ is an *entire function of type II* if its canonical representation is of the form

$$\Psi(z) = Cz^n e^{-\gamma z^2 + \delta z} \prod_{\nu=1}^{\infty} (1+\delta_\nu z)\, e^{-\delta_\nu z}, \quad (C \lesseqgtr 0, n \geqq 0,\ \gamma \geqq 0,\ \gamma, \delta, \text{ real}). \tag{9}$$

It was shown by Laguerre and Polya[4] that the entire functions of type I *and no others* are the uniform limits ($\not\equiv 0$) of real polynomials having all their roots on the half-line $x \leqq 0 (z = x + yi)$. Likewise that the entire functions of type II *and no others* are the uniform limits ($\not\equiv 0$) of real polynomials with only real roots. The coefficients of the power series expansions of the functions $\Phi(z)$ and $\Psi(z)$ about the origin enjoy remarkable algebraic properties which were discovered by Polya and Schur (*loc. cit.*) on the foundation of Laguerre's pioneering work. Finally, Polya[5] investigated the expansion coefficients of the reciprocals $1/\Phi(z)$ and $1/\Psi(z)$ about the origin. His results suggested that these reciprocals should allow of representations by the ordinary Laplace-Stieltjes integral and by the bilateral Laplace-Stieltjes integral, respectively, both with monotone determining functions. This problem was later investigated by H. Hamburger.[6] However, Hamburger's results do not exhibit the exact nature of the Laplace integral representations of these particular classes of functions.

3. *Laplace integrals.*—The relationship between the reciprocals of functions of the Laguerre-Polya-Schur type and the Laplace integral is described by the following theorem.

THEOREM 1. *Let* $\Lambda(x)$ *be a totally positive function which is not of the form* $\exp\{ax + b\}$. *Then the bilateral Laplace integral*

$$\int_{-\infty}^{\infty} e^{xz} \Lambda(x) dx \tag{10}$$

converges in a strip $\alpha < Rz < \beta (-\infty \leqq \alpha < \beta \leqq +\infty)$ *and represents there the reciprocal*

$$\frac{1}{\Psi(z)} \tag{11}$$

of an entire function $\Psi(z)$ of type II which is not of the form $C \cdot e^{\delta z}$. The endpoints α, β of the strip of convergence of the integral (10) are zeros of $\Psi(z)$, provided they are finite.

Conversely, let (11) be the reciprocal of a function (9) of type II, not of the form $C \cdot e^{\delta z}$, and let $\alpha < Rz < \beta (-\infty \leqq \alpha < \beta \leqq +\infty)$ be a strip in which (11) is regular. Then $1/\Psi(z)$ (if $\Psi(z) > 0$ in $\alpha < z < \beta$), or else $-1/\Psi(z)$ (if $\Psi(z) < 0$ in $\alpha < z < \beta$), may be there represented by a Laplace integral (10), where $\Lambda(x)$ is a totally positive function.

We, therefore, have a 1-1 correspondence between t. p. functions and reciprocals of functions of type II in a given strip of regularity $\alpha < Rz < \beta$ $(-\infty \leqq \alpha < \beta \leqq \infty)$ by the relation

$$\int_{-\infty}^{\infty} e^{xz} \Lambda(x) dx = \frac{1}{\Psi(z)} \cdot \epsilon, \qquad (\alpha < Rz < \beta), \tag{12}$$

where $\epsilon = sgn \Psi(z)$ in $\alpha < z < \beta$.

Further properties of the correspondence (12) are as follows.

THEOREM 2. *The totally positive function $\Lambda(x)$ is a Polya frequency function if and only if*

$$\alpha < 0 < \beta \tag{13}$$

Otherwise $\Lambda(x)$ is always monotone, namely, non-increasing if $0 \leqq \alpha$ and non-decreasing if $\beta \leqq 0$.

THEOREM 3. *The relation (12) may always be inverted by Mellin's integral to*

$$\Lambda(x) = \frac{1}{2\pi i} \int_{\sigma - \infty i}^{\sigma + \infty i} \frac{\epsilon}{\Psi(z)} e^{-xz} dx, \quad (\alpha < \sigma < \beta, -\infty < x < \infty). \tag{14}$$

Unless $\Psi(z)$ is free of zeros, there are several t. p. functions $\Lambda(x)$ associated with the same $\Psi(z)$, corresponding to the various strips of regularity of its reciprocal. The way in which these various $\Lambda(x)$ are connected with each other is as follows.

THEOREM 4. *Let (α, β) and (β, γ) be contiguous intervals of regularity of $1/\Psi(z)$ $(-\infty \leqq \alpha < \beta < \gamma \leqq \infty)$ and let $\Lambda(x)$ and $\Lambda_1(x)$ be the corresponding totally positive functions, i.e.,*

$$\frac{\epsilon}{\Psi(z)} = \int_{-\infty}^{\infty} e^{xz} \Lambda(x) dx \ (\alpha < Rz < \beta), \quad \frac{\epsilon_1}{\Psi(z)} = \int_{-\infty}^{\infty} e^{xz} \Lambda_1(x) dx \quad (\beta < Rz < \gamma),$$

where $\epsilon = \pm 1$, $\epsilon_1 = \pm 1$, such that $\epsilon \Psi(z) > 0$ if $\alpha < z < \beta$, and $\epsilon_1 \Psi(z) > 0$ if $\beta < z < \gamma$. Then

$$\epsilon_1\Lambda_1(x) - \epsilon\Lambda(x) = \text{The residue of } e^{-xz}/\Psi(z) \text{ at } z = \beta, \qquad (15)$$

for all real values of x.

Functions $\Phi(z)$ of type I being at the same time also functions of type II, the theorems just stated apply to such functions as well. In this particular case, however, we may add to Theorem 1 the following additional information.

THEOREM 5. *Let* $\Lambda(x)$ *be a Polya frequency function such that*

$$\Lambda(x) = 0 \text{ if } x > 0. \qquad (16)$$

Then

$$\int_{-\infty}^{0} e^{xz}\Lambda(x)dx = \frac{1}{\Phi(z)}, \ (Rz > \alpha, \text{ for some } \alpha < 0) \qquad (17)$$

where $\Phi(z)$ *is an entire function of type I, with* $\Phi(0) > 0$, *and not of the form* $C \cdot e^{bz}$. *Conversely, if* $\Phi(z)$ *is of type I,* $\Phi(0) > 0$, *not of the form,* $C \cdot e^{bz}$, *then it may be represented by the ordinary Laplace integral (17), where* $\Lambda(x)$, *if defined as* $= 0$ *for* $x > 0$, *is a Polya frequency function.*

4. *Examples.*—A few classical integral representations will serve to illustrate the theory. The following formulae are due to Euler:

$$\int_{-\infty}^{\infty} e^{xz}e^{-e^x}dx = \Gamma(z) \qquad (Rz > 0), \qquad (18)$$

$$\int_{-\infty}^{\infty} e^{xz}\frac{1}{1+e^x}dx = \frac{\pi}{\sin \pi z} \qquad (0 < Rz < 1), \qquad (19)$$

$$\int_{-\infty}^{0} e^{xz}(1-e^x)^{n-1}e^x dx = \frac{(n-1)!}{(z+1)(z+2)\ldots(z+n)} \quad (Rz > -1). \qquad (20)$$

The right-hand sides are reciprocals of functions of type II, hence the determining functions $\Lambda(x)$ are all totally positive by Theorem 1[7] (converse part). The direct part of Theorem 1 may at times also be applied and that should be the direction of the more significant applications, if any. To illustrate this point, let us prove (the well-known fact) that $1/\Gamma(z)$ is of type II. By (18) it is sufficient to show that $\Lambda(x) = \exp\{-e^x\}$ is totally positive. But this is clear since

$$\det\|\Lambda(x_i - t_j)\| = \det\|\exp\{-e^{x_i}\cdot e^{-t_j}\}\| = \det\|\exp\{\alpha_i\beta_j\}\| > 0,$$

since both sequences of numbers

$$\alpha_i = e^{x_i}, \ \beta_i = -e^{-t_i} \qquad (i = 1, 2, \ldots, n)$$

are monotone increasing. A similar, equally simple argument applies to (19) showing that $\sin(\pi z)$ is of type II.

In order to illustrate Theorem 4 we inquire into the integral representation

$$\int_{-\infty}^{\infty} e^{xz}\Lambda_n(x)dx = (-1)^{n+1}\Gamma(z), (-n-1 < Rz < -n), (n > 0), \quad (21)$$

which is assured by Theorem 1. Starting from the representation (18), the various transitional residues appearing in (15) are readily determined with the result that

$$\Lambda_n(x) = (-1)^{n+1}\{e^{-e^x} - (1 - \frac{1}{1!}e^x + \frac{1}{2!}e^{2x} - \ldots \pm \frac{1}{n!}e^{nx})\},$$
$$(-\infty < x < \infty). \quad (22)$$

This function is totally positive by Theorem 1.

Formula (20) illustrates Theorem 5. Another example to Theorem 5 is furnished by the formula

$$\int_0^{\infty} e^{xr}\,dL(x) = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{z}{n^2}}, \qquad (Rz < 1), \qquad (23)$$

where the distribution function $L(x)$ is defined by

$$L(x) = \vartheta_0\left(0 \middle| \frac{ix}{\pi}\right) = \sum_{n=-\infty}^{\infty}{}' (-1)^n e^{-xn^2}, \qquad (x > 0, L(0) = 0).$$

Its derivative

$$\Lambda(x) = \begin{cases} \sum_{-\infty}^{\infty}{}' (-1)^{n+1}n^2 e^{-xn^2} & \text{if } x > 0, \\ 0 & \text{if } x \leqq 0, \end{cases} \qquad (24)$$

is, by Theorem 5, a Polya frequency function.

5. *Concluding remarks.*—1. Various other properties of a t. p. function $\Lambda(x)$ are reflected in corresponding properties of the associated function $\Psi(z)$ of (12). Thus $\Lambda(x)$ is of class C^{∞} if, and only if, in (9) we have either $\gamma > 0$, or else $\gamma = 0$ and $\delta_\nu > 0$ for infinitely many ν. Thus the $\Lambda(x)$ of (24) is of class C^{∞}, i.e., all derivatives of $\Lambda(x)$ vanish at the origin. However, if $\gamma = 0$ and $\Psi(z)$ is a polynomial of degree $m(> 0)$ multiplied by an exponential factor, then $\Lambda(x)$ is exactly of class C^{m-2}. Thus the $\Lambda(x)$ of (20) has exactly $n - 2$ continuous derivatives.

2. It is expected that Polya frequency functions will be useful in descriptive statistics for the purpose of curve-fitting to empirical statistical data. One of the reasons for expecting such use is the extreme smoothness of these curves as a whole. By this we mean the following property: Let $\Lambda(x)$ be a Polya frequency function of class C^{∞}. By a repeated application of Rolle's theorem we conclude that $\Lambda^{(n)}(x)$ has *at least* n distinct real zeros. Actually $\Lambda^{(n)}(x)$ *has exactly* n *simple real zeros and this is true for all values of* n. An approach to Polya frequency functions from an entirely different

point of view will be discussed in a joint paper of H. B. Curry and the author.

3. A proof of Theorem 1 is essentially based on the results and methods developed by Polya and Schur.[7] The only additional element required is a set of sufficient conditions insuring that a linear transformation be variation-diminishing.[8] Such conditions will serve to establish the following essential lemma: *If $\Lambda(x)$ is a Polya frequency function and $t_1 < t_2 < \ldots < t_n$, then the linear combination*

$$F(x) = a_1\Lambda(x - t_1) + a_2\Lambda(x - t_2) + \ldots + a_n\Lambda(x - t_n)$$

obeys the rule of signs of Descartes, i.e., the number of variations of sign of $F(x)$, for all real x, cannot exceed the number of variations of signs in the sequence $a_1, a_2, \ldots, a_n$, of its coefficients.

4. The discreet analogue of a t. p. function is a totally positive sequence $\{a_n\}$ $(n = 0, \pm 1, \pm 2, \ldots)$ with the property that the 4-way infinite matrix $||a_{i-j}||$ has only non-negative minors. The author expects to discuss the nature and properties of such sequences on a future occasion.

[1] Determinants related to the one appearing here were considered by A. Bloch and G. Polya, "Abschätzungen des Betrages einer Determinante," *Vierteljahrschrift Zürich*, **78**, 27–33 (1933).

[2] Polya, G., and Szegö, G., *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Problem 76, p. 49.

[3] Polya, G., and Schur, I., "Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen," *Journal für Math.*, **144**, 89–113 (1914), especially p. 93.

[4] Polya, G., "Über Annäherung durch Polynome mit lauter reellen Wurzeln," *Rendiconti di Palermo*, **36**, 1–17 (1913). Laguerre assumes uniform convergence in every finite domain. Polya assumes such convergence only in a neighborhood of the origin.

[5] Polya, G., "Algebraische Untersuchungen über ganze Funktionen vom Geschlechte Null und Eins," *Journal für Math.*, **145**, 224–249 (1915).

[6] Hamburger, H., "Bemerkungen zu einer Fragestellung des Herrn Polya," *Math. Z.*, **7**, 302–322 (1920).

[7] We are actually using here the following addition to Theorem 1: If the reciprocal of a function of type II is represented by a Laplace integral (12), where $\Lambda(x)$ is *continuous*, then $\Lambda(x)$ is totally positive.

[8] Schoenberg, I., "Über variationsvermindernde lineare Transformationen," *Math. Z.*, **32**, 321–328 (1930), especially Satz 1.

ON THE LOCATION OF THE CRITICAL POINTS OF HARMONIC MEASURE

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The object of this note is the proof of

THEOREM 1. *Let C be the unit circle in the z-plane, and let closed arcs α_k $(k = 1,2, \ldots, n)$ of C be mutually disjoint. Denote by $z = a_k$ and $z = b_k$ the initial and terminal points of α_k, with notation so chosen that the points a_k and b_k are ordered positively (i.e., counter-clockwise) on C as $a_1, b_1, a_2, b_2, \ldots, a_n b_n$, $a_1 = a_{n+1}$, $b_1 = b_{n+1}$. Denote by $u(z)$ the harmonic measure with respect to the interior of C of the set $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ at a point z interior to C; thus $u(z)$ is harmonic and bounded interior to C, and approaches unity as z approaches an interior point of an arc α_k, and approaches zero as z approaches an interior point of the arcs complementary to the set α. Then no critical points of $u(z)$ lie in the region R_k [or S_k] interior to C bounded by the arc A_k: $a_k b_k a_{k+1}$ [or B_k: $b_k a_{k+1} b_{k+1}$] of C and the arc A_k': $a_k a_{k+1}$ [or B_k': $b_k b_{k+1}$] of a circle orthogonal to C. No critical point of $u(z)$ lies on A_k' [or B_k'] unless $n = 2$, in which case the unique critical point of $u(z)$ lies at the intersection of A_1' and B_1'.*

Thus a point z interior to C cannot be a critical point of $u(z)$ if a circle orthogonal to C separates z and one of the points a_k, b_k from all the other points a_j, b_j.

In the case $n > 2$, Theorem 1 enables us readily to construct a polygon of $2n$ sides interior to C bounded by arcs of non-euclidean straight lines which contains in its interior all the $n-1$ critical points of $u(z)$ interior to C.

It is to be expected that the arcs A_k' and B_k' should play similar rôles in Theorem 1, for the harmonic measure with respect to the interior of C of the complement of the set α in the point z is $1 - u(z)$, which has the same critical points as $u(z)$. In the proof of Theorem 1, we restrict ourselves to the case of R_k and the arcs A_k and A_k', as is sufficient.

The harmonic measure of the arc α_k with respect to C in the point z is readily computed to be

$$\frac{1}{\pi}[\arg(z - b_k) - \arg(z - a_k) - (1/2)\alpha_k], \tag{1}$$

where the arguments are suitably chosen, and where α_k represents the angular measure of the arc α_k. The function (1) is the real part of the analytic function

$$\frac{1}{\pi i}\left[\log(z - b_k) - \log(z - a_k) - \frac{i\alpha_k}{2}\right]. \tag{2}$$

The critical points of $u(z)$ are then the zeros of the function $\varphi'(z)$, where $\varphi(z)$ is the sum of the functions (2) for all k; we have

$$i\varphi'(z) = \frac{1}{\pi}\sum_{k=1}^{n}\left[\frac{1}{z - b_k} - \frac{1}{z - a_k}\right]. \tag{3}$$

In (3) we take the conjugate of each term. The quantity $1/(\bar{z} - \bar{b}_k)$ represents a vector of magnitude $1/|z - b_k|$ whose direction and sense are those of the vector $(z - b_k)$. It follows that *the zeros of $\varphi'(z)$ are the positions of equilibrium in the field of force due to unit positive particles situated at each point b_k, and unit negative particles situated at each point a_k, where each positive particle repels and each negative particle attracts, with a force equal to the inverse distance.*

The total force at the center 0 of C can be interpreted in magnitude, direction and sense as the sum of all the vectors b_k0 and $0a_k$, or as the sum of all the vectors b_ka_k. Moreover, in any conformal map of the interior of C onto itself the function $u(z)$ is invariant, as are the critical points of $u(z)$, so we proceed to study those critical points by transforming C into itself so that the point to be studied is transformed into 0.

Let z_0 denote an arbitrary point interior to R_k or on the arc A_k'; we transform z_0 into 0, $1/\bar{z}_0$ into the point at infinity, and b_k into the point $z = 1$, by a linear transformation of z; we retain the original notation. The point $z_0 = 0$ lies in R_k or on A_k' so the arc A_k has angular measure at least π, and has angular measure π if and only if z_0 lies on A_k'. The positive arc $(a_{k+1}, -1)$ is not greater than the positive arc $(a_k, +1)$.

If $z_0 = 0$ is a position of equilibrium in the field of force previously considered, the algebraic sum of the vectors b_ja_j must vanish. However, the point a_{k+1} cannot lie interior to the positive arc $(b_k, -a_k)$, and the points $b_{k+1}, a_{k+2}, \ldots, b_n, a_1, \ldots, b_{k-1}$ lie on the positive arc $a_{k+1}a_k$, from which it follows that the sum of the horizontal components in the positive horizontal direction of the vectors $b_{k+1}a_{k+1}, b_{k+2}a_{k+2}, \ldots, b_na_n, b_1a_1 \ldots, b_{k-1}a_{k-1}$ is algebraically less than the magnitude of the horizontal component of the vector b_ka_k (which is negative), except in the special case $n = 2$, $a_1 = -a_2$, $b_1 = -b_2$. Thus the total force at $z_0 = 0$ cannot be zero, and $z_0 = 0$ cannot be a critical point of $u(z)$ except in this special case, and the theorem is established.

Theorem 1 is remarkable in the fact that for $n > 2$ it exhibits a polygonal region containing all critical points, and yet a degenerate case of the region is the unique critical point in a non-trivial situation ($n = 2$).

Theorem 1 can be interpreted as referring to the zeros of the derivative (or logarithmic derivative) of an arbitrary rational function

$$\prod_{k=1}^{n}\left(\frac{z - b_k}{z - a_k}\right)$$

whose poles and zeros are all simple and finite, and which alternate (i.e., are interlaced) on an arbitrary circle C of the extended plane. From this standpoint, the finite zeros of the derivative lie in two curvilinear polygons; the polygons lie one in each of the two regions into which C separates the plane; polygons and finite zeros are symmetric in C. The point at infinity is also a zero of the derivative.

Theorem 1 can obviously be extended by conformal mapping:

THEOREM 2. *Let C be an arbitrary Jordan curve, and let closed arcs α_k $(k = 1, 2, \ldots, n)$ of C be mutually disjoint. We denote by a_k and b_k the initial and terminal points of α_k, with notation so chosen that the points a_k and b_k are ordered counter-clockwise on C as $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$, $a_1 = a_{n+1}, b_1 = b_{n+1}$. Denote by $u(z)$ the harmonic measure with respect to the interior of C of the set $\alpha_1 + \alpha_2 + \ldots + \alpha_n$ at a point z interior to C. Then no critical points of $u(z)$ lie in the region R_k [or S_k] bounded by the arc A_k: $a_kb_ka_{k+1}$ [or B_k:$b_ka_{k+1}b_{k+1}$] of C and the arc A_k': a_ka_{k+1} [or B_k':b_kb_{k+1}] of a non-euclidean straight line for the interior of C. No critical point of $u(z)$ lies on A_k' unless $n = 2$, in which case the unique critical point of $u(z)$ lies at the intersection of A_1' and B_1'.*

It may be noted that in Theorems 1 and 2 the arc A_k' [or B_k'] is the locus of points z at which the harmonic measure of the arc A_k [or B_k] of C with respect to the interior of C has the value one-half. Thus the conclusion of Theorems 1 and 2 may be expressed by asserting that *at a critical point z of $u(z)$ interior to C, the harmonic measure of every arc A_k [or B_k] with respect to the interior of C has a value less than one-half, except in the case $n = 2$, when this value equals one-half.*

Theorem 1 extends to the case of an infinite number of arcs α_k and as thus extended applies in the study of critical points of harmonic measures in multiply connected regions, by a conformal map of their universal covering surfaces.

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*THEORY OF HARMONIC TRANSFORMATIONS**

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1. Consider a transformation T of the plane

$$X = \phi(x, y), \quad Y = \psi(x, y), \tag{1}$$

with the Jacobian $J = \phi_x\psi_y - \phi_y\psi_x \neq 0$, such that the components satisfy the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0, \psi_{xx} + \psi_{yy} = 0. \tag{2}$$

We shall term any such correspondence T a *harmonic transformation.*

Harmonic correspondences should not be confused with conformal maps. In general, the components of a harmonic transformation are not interrelated in any way whatsoever. The components ϕ and ψ of a conformal correspondence are of course conjugate-harmonic, that is, they satisfy the direct or reverse Cauchy-Riemann equations

$$\phi_x = \pm\psi_y, \phi_y = \pm\psi_x. \tag{3}$$

Thus a harmonic transformation is conformal if and only if its components are conjugate-harmonic.

All harmonic correspondences form an infinite set (H) of $\infty^{4f(1)}$ transformations since they are defined by essentially four independent functions of a single variable. Our infinite set (H) contains as a subgroup the $\infty^{2f(1)}$ conformalities which are defined by two independent functions of a single variable. The totality (H) of harmonic transformations, of course, do not constitute a group.

General harmonic transformations are of interest in connection with the theory of minimal surfaces and the Plateau problem. See the fundamental papers of Schwarz and Douglas. See footnote (5) for references.

We shall develop the theory of harmonic transformations from four points of view. See footnotes 1, 2 and 3. Firstly, we shall obtain various characterizations of the infinite set (H) of harmonic transformations by means of isothermal properties. From this, we deduce new characterizations of the conformal group. Secondly, we shall consider the groups contained in the infinite set (H). Thirdly, we obtain various properties of (H) by means of the derivatives of a polygenic function. Fourthly, and finally, we shall give a discussion of the transformation theory of differential elements of first, second, and third orders (at a fixed point) induced by the infinite set (H).

2. *If by a transformation T from the (x, y)-plane to the (X, Y)-plane, more than four distinct parallel pencils of straight lines in the (X, Y)-plane correspond to isothermal families in the (x, y)-plane, then in the real domain, either T is a harmonic transformation; or else T is the product of a conformality by a circle-to-line transformation. In all cases every parallel pencil of straight lines in the (X, Y)-plane corresponds to an isothermal family in the (x, y)-plane.*

If the restriction of the pencils of straight lines being parallel is removed, we obtain the following result.

If by a transformation T from the (x, y)-plane to the (X, Y)-plane, every pencil (parallel or not) of straight lines in the (X, Y)-plane corresponds to an isothermal family in the (x, y)-plane, then T is either a conformal transforma-

tion, or a circle-to-line correspondence, or the product of a conformality by a circle-to-line transformation.

3. By considering not only parallel pencils of straight lines but also concentric sets of circles, we derive the proposition:

The only transformations in the real domain whereby every parallel pencil of straight lines and also every concentric set of circles correspond to isothermal families of curves are the conformalities.

For the imaginary domain, the corresponding proposition is the following one.

The only transformations in the real or imaginary domain whereby every pencil of circles corresponds to an isothermal family are the conformal maps.

4. In general, the inverse of a harmonic transformation is not harmonic.[6] We obtain all harmonic correspondences with harmonic inverses. The following result is established.

In the real domain, the complete set of harmonic transformations whose inverses are also harmonic consists of those of the infinite conformal group, those of the six-parameter affine group, and those of the eight-parameter set of transformations

$$mU = \log \frac{ae^{nu} + b}{A + Be^{-Nv}}, \; MV = \log \frac{Ae^{Nv} + B}{a + be^{-nu}}, \tag{4}$$

where (A, B, M, N) *are the conjugates of the constants* (a, b, m, n), *and where* $m \neq O$, $M \neq O$, $n \neq 0$, $N \neq O$, *and* $aA - bB \neq O$.

Here (u, v) denote minimal coordinates of a point, that is, $u = x + iy$ and $v = x - iy$.

The inverse of any transformation of the eight-parameter set (4) is in the set. If $S(a, b, m, n)$ is any transformation of this set, its inverse is $S(A, -b, n, m)$. The identity is given by $a = A$, $m = n$, $b = O$.

However this set (4) does not constitute a group since the product of any two transformations of this set (4) is not in the set (4) in general.

In the imaginary domain, it is found that there are four other sets of harmonic transformations with harmonic inverses in addition to the three sets stated above.

5. *In the imaginary domain, the only groups of harmonic transformations are the infinite conformal group, the six-parameter affine group, the two infinite groups*

$$\begin{aligned} U &= au + g(v), \; V = F(v), \; a \neq O, \; F_v \neq O; \\ U &= f(u), \; V = av + G(u), \; a \neq O, \; f_u \neq O; \end{aligned} \tag{5}$$

and the subgroups of these four groups.

In the real domain, the only harmonic groups are the conformal group, the affine group, and the subgroups of these two.

6. *A point transformation T is harmonic if and only if the center trans-*

formation induced by the first derivative of the related polygenic function $w = \phi + i\psi$ *is direct conformal.*

We find other properties of harmonic transformations with reference to the second derivative of the polygenic function. One such result is the following.

If a transformation T is harmonic, then the fundamental conic section Q is a horizontal parabola[4].

7. We consider the transformation theory of differential elements of first, second and third orders (at a fixed point) induced by the infinite set (H) of harmonic transformations. For the cases of first and second order differential elements, this transformation theory for the harmonic case is identical with that induced by the total group of all point transformations. But for third order differential elements, we establish the following proposition.

The infinite set (H) *of harmonic transformations induces on the differential elements of third order a twelve-parameter set* S_{12} *of transformations, which is a subset of the fifteen-parameter group* G_{15} *induced by the group of all-point transformations.*

Elsewhere we shall develop a characterization of this twelve-parameter set of transformations S_{12} of third order differential elements.

* Presented to the American Mathematical Society, 1946.

The following papers constitute a short bibliography of the related subjects developed by Kasner and jointly by Kasner and De Cicco.

1. "The Geometry of Differential Elements of Second Order with Respect to the Group of All Point Transformations," *Amer. Journ. Math.*, **28**, 203–213 (1906).

2. "Biharmonic Functions and Certain Generalizations," *Amer. Journ. Math.*, **58**, 377–390 (1936).

3. "The Pseudo-Angle in Space of $2n$-Dimensions"; also "Bi-Isothermal Systems," *Bull. Amer. Math. Soc.*, **51**, 162–174 (1945). "Circle-to-line Transformations," *Amer. Math. Monthly*, **52**, 425–433, (1945).

4. "The Geometry of Polygenic Functions," *Revista Universidad Tucuman, Argentina*, **4**, 7–45 (1944). "The Second Derivative of a Polygenic Function," *Trans. Amer. Math. Soc.* **30**, 803–818 (1928).

5. Schwarz, *Abhandlungen*, Vol. 1, p. 293. Also see abstracts by Douglas in *Bull. Amer. Math. Soc.* (1943).

6. Douglas also has studied the inversion of harmonic transformations. See *Bull. Amer. Math. Soc.* (1944), p. 180.